ON THE DISTRIBUTION OF NONTRIVIAL ZEROS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. The distribution of nontrivial zeros of the Riemann zeta function was investigated in this paper. First, a curve integral of the Riemann zeta function $\zeta(s)$ was formed, which is along a horizontal line from s to $1-\bar{s}$ which are two nontrivial zeros of $\zeta(s)$ and symmetric about the vertical line $\sigma=\frac{1}{2}$. Next, the result of the curve integral was derived and proved equal to zero. Then, by proving a lemma of central dissymmetry of the Riemann zeta function $\zeta(s)$, two nontrivial zeros s and $1-\bar{s}$ were proved being a same zero or satisfying $s=1-\bar{s}$. Hence, nontrivial zeros of $\zeta(s)$ all have real part $\Re(s)=\frac{1}{2}$.

Part 1. On the distribution of nontrivial zeros of the Riemann zeta function by using the series representation of $\zeta(s)$

INTRODUCTION

It is well known that the Riemann zeta function $\zeta(s)$ of a complex variable $s=\sigma+it$ is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

for the real part $\Re(s)>1$ and its analytic continuation in the half plane $\sigma>0$ is

$$\zeta(s)=\sum_{n=1}^{N}\frac{1}{n^s}-\frac{N^{1-s}}{1-s}-\frac{1}{2}N^{-s}+s\int_{N}^{\infty}\frac{\frac{1}{2}-\{x\}}{x^{s+1}}dx \tag{0.1}$$

for any integer $N\geq 1$ and $\Re(s)>0$[8-10]. It extends to an analytic function in the whole complex plane except for having a simple pole at $s=1$. Trivially, $\zeta(-2n)=0$ for all positive integers. All other zeros of the Riemann zeta functions are called its nontrivial zeros[1-10].

The Riemann hypothesis states that nontrivial zeros of $\zeta(s)$ all have real part $\Re(s)=\frac{1}{2}$.

The investigation of the distribution of nontrivial zeros of the Riemann zeta function in this paper includes several parts. First, a curve integral of the Riemann zeta function $\zeta(s)$ was formed, which is along a horizontal line from s to $1-\bar{s}$ which are two nontrivial zeros of $\zeta(s)$ and symmetric about the vertical line $\sigma=\frac{1}{2}$. Next, the result of the curve integral was derived and proved equal to zero. Then, by proving a lemma of central dissymmetry of the Riemann zeta function $\zeta(s)$, two

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nontrivial zeros s and $1-\bar{s}$ were proved being a same zero or satisfying $s=1-\bar{s}$. Hence, nontrivial zeros of $\zeta(s)$ all have real part $\Re(s)=\frac{1}{2}$.

The distribution of nontrivial zeros of the Riemann zeta function was also investigated in the author's another paper by using another representation of Formula (0.1) (See representation (2.1.5) on page 14 of the reference book[8])

$$\zeta(s)=s\int_0^\infty \frac{[x]-x}{x^{s+1}}dx, 0<\sigma<1. \tag{0.2}$$

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1. Curve integral

The Riemann zeta function $\zeta(s)$ defined in Formula (0.1) is analytic in a simply connected complex region D. According to the Cauchy theorem, suppose that C is any simple closed curve in D, then, there is

$$\oint_C \zeta(s)ds=0. \tag{1.1}$$

According to the theory of the Riemann zeta function[8-10], nontrivial zeros of $\zeta(s)$ are symmetric about the vertical line $\sigma=\frac{1}{2}$. When a complex number $s_0=\sigma_0+it_0$ is a nontrivial zero of $\zeta(s)$, the complex number $1-\sigma_0+it_0=1-\bar{s}_0$ is also a nontrivial zero of $\zeta(s)$, and there must be $\zeta(s_0)=0$ and $\zeta(1-\bar{s}_0)=0$.

Let consider the curve integral of $\zeta(s)$ along a horizontal line from s_0 to $1-\bar{s}_0$ as following

$$\int_{s_0}^{1-\bar{s}_0}\zeta(s)ds=F(1-\bar{s}_0)-F(s_0) \texttt{ for } 0<\sigma_0\le\frac{1}{2} \tag{1.2}$$

where $F(s)$ is the primitive function of $\zeta(s)$.

By using Formula (0.1), Equation (1.2) can be written as

$$\begin{aligned} F(1-\bar{s}_0)-F(s_0) &= \int_{s_0}^{1-\bar{s}_0}\zeta(s)ds \\ &= \sum_{n=1}^N\int_{s_0}^{1-\bar{s}_0}\frac{1}{n^s}ds-\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{1-s}ds \\ &-\frac{1}{2}\int_{s_0}^{1-\bar{s}_0}N^{-s}ds+\int_{s_0}^{1-\bar{s}_0}s\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{s+1}}dxds. \end{aligned} \tag{1.3}$$

1.1. **Integral I.**
The first integral on the right hand side of Equation (1.3) can be estimated as follows:

$$|\sum_{n=1}^N\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}|\le\sum_{n=1}^N\int_{\sigma_0}^{1-\sigma_0}\frac{d\sigma}{n^\sigma}=\int_{\sigma_0}^{1-\sigma_0}\sum_{n=1}^N\frac{d\sigma}{n^\sigma}\le\int_{\sigma_0}^{1-\sigma_0}\int_0^N\frac{dx}{x^\sigma}d\sigma$$

$$=\int_{\sigma_0}^{1-\sigma_0}\frac{N^{1-\sigma}}{1-\sigma}d\sigma\le\int_{\sigma_0}^{1-\sigma_0}\frac{N^{1-\sigma}}{\sigma_0}d\sigma=\frac{N^{1-\sigma_0}-N^{\sigma_0}}{\sigma_0\log N}.$$

Thus, we have

$$|\sum_{n=1}^N\int_{s_0}^{1-\bar{s}_0}\frac{1}{n^s}ds|\le\frac{N^{1-\sigma_0}-N^{\sigma_0}}{\sigma_0\log N}. \tag{1.4}$$

1.2. **Integral II.** The second integral on the right hand side of Equation (1.3) is

$$\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{1-s}ds = -\frac{1}{\log N}\frac{N^{1-s}}{1-s}|_{s_0}^{1-\bar{s}_0} + \frac{1}{\log N}\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^2}ds$$

$$= -\frac{1}{\log N}\frac{N^{1-s}}{1-s}|_{s_0}^{1-\bar{s}_0} - \frac{1!}{\log^2 N}\frac{N^{1-s}}{(1-s)^2}|_{s_0}^{1-\bar{s}_0} + \frac{2!}{\log^2 N}\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^3}ds$$

$$= \cdots = -\sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}\frac{N^{1-s}}{(1-s)^m}|_{s_0}^{1-\bar{s}_0} + \frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+1}}ds.$$

In the notice of $\zeta(s_0) = 0$ and $\zeta(1-\bar{s}_0) = 0$, by using Formula (0.1), there are

$$\sum_{n=1}^{N} \frac{1}{n^{s_0}} - \frac{N^{1-s_0}}{1-s_0} - \frac{1}{2}N^{-s_0} + s_0\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx = 0$$

and

$$\sum_{n=1}^{N} \frac{1}{n^{1-\bar{s}_0}} - \frac{N^{\bar{s}_0}}{\bar{s}_0} - \frac{1}{2}N^{-(1-\bar{s}_0)} + (1-\bar{s}_0)\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx = 0.$$

Thus, we have

$$-\frac{N^{1-s}}{(1-s)^m}|_{s_0}^{1-\bar{s}_0} = \frac{N^{1-s_0}}{(1-s_0)^m} - \frac{N^{\bar{s}_0}}{\bar{s}_0^m}$$

$$= \frac{1}{(1-s_0)^{m-1}}[\sum_{n=1}^{N} \frac{1}{n^{s_0}} - \frac{1}{2}N^{-s_0} + s_0\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx]$$

$$-\frac{1}{\bar{s}_0^{m-1}}[\sum_{n=1}^{N} \frac{1}{n^{1-\bar{s}_0}} - \frac{1}{2}N^{-(1-\bar{s}_0)} + (1-\bar{s}_0)\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx]$$

$$= \frac{1}{(1-s_0)^{m-1}}\sum_{n=1}^{N} \frac{1}{n^{s_0}} - \frac{1}{\bar{s}_0^{m-1}}\sum_{n=1}^{N} \frac{1}{n^{1-\bar{s}_0}} - \frac{1}{2}[\frac{N^{-s_0}}{(1-s_0)^{m-1}} - \frac{N^{-(1-\bar{s}_0)}}{\bar{s}_0^{m-1}}]$$

$$+\frac{s_0}{(1-s_0)^{m-1}}\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx - \frac{1-\bar{s}_0}{\bar{s}_0^{m-1}}\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx$$

and

$$\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{1-s}ds = \sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\sum_{n=1}^{N} \frac{1}{n^{s_0}} - \frac{1}{\bar{s}_0^{m-1}}\sum_{n=1}^{N} \frac{1}{n^{1-\bar{s}_0}}] \tag{1.5}$$

$$-\frac{1}{2}\sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}[\frac{N^{-s_0}}{(1-s_0)^{m-1}} - \frac{N^{-(1-\bar{s}_0)}}{\bar{s}_0^{m-1}}]$$

$$+\sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}[\frac{s_0}{(1-s_0)^{m-1}}\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx - \frac{1-\bar{s}_0}{\bar{s}_0^{m-1}}\int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx]$$

$$+\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+1}}ds.$$

1.3. **Integral III.** The third integral on the right hand side of Equation (1.3) is

$$\int_{s_0}^{1-\bar{s}_0} N^{-s}ds = -\frac{1}{\log N}N^{-s}|_{s_0}^{1-\bar{s}_0} = \frac{1}{\log N}(\frac{1}{N^{s_0}} - \frac{1}{N^{1-\bar{s}_0}})$$

1.4. **Integral results.** Thus, we get

$$
\begin{aligned}
F(1-\bar{s}_0)-F(s_0) &= \int_{s_0}^{1-\bar{s}_0}\zeta(s)ds = \sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s} \\
&-\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}] \\
&+\frac{1}{2}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{N^{-s_0}}{(1-s_0)^{m-1}}-\frac{N^{-(1-\bar{s}_0)}}{\bar{s}_0^{m-1}}] \\
&-\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{s_0}{(1-s_0)^{m-1}}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx-\frac{1-\bar{s}_0}{\bar{s}_0^{m-1}}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx] \\
&-\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds \\
&-\frac{1}{2}\frac{1}{\log N}(\frac{1}{N^{s_0}}-\frac{1}{N^{1-\bar{s}_0}}).+\int_{s_0}^{1-\bar{s}_0}s\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{s+1}}dxds \\
&= f_1(N,M)+f_2(N,M)+f_3(N,M)+f_4(N,M)
\end{aligned}
$$

and

$$
\begin{aligned}
|F(1-\bar{s}_0)-F(s_0)| &= |\int_{s_0}^{1-\bar{s}_0}\zeta(s)ds| \\
&\le |f_1(N,M)|+|f_2(N,M)|+|f_3(N,M)|+|f_4(N,M)|
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(N,M) &= \sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}-\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds \\
&-\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}],
\end{aligned}
\tag{1.6}
$$

$$
\begin{aligned}
f_2(N,M) &= -\frac{1}{2}\frac{1}{\log N}[1-\sum_{m=1}^{M}\frac{(m-1)!}{\log^{m-1}N}\frac{1}{(1-s_0)^{m-1}}]\frac{1}{N^{s_0}} \\
&+\frac{1}{2}\frac{1}{\log N}[1-\sum_{m=1}^{M}\frac{(m-1)!}{\log^{m-1}N}\frac{1}{\bar{s}_0^{m-1}}]\frac{1}{N^{1-\bar{s}_0}} \\
&= \frac{1}{2}\sum_{m=2}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{N^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{N^{1-\bar{s}_0}}],
\end{aligned}
\tag{1.7}
$$

$$
\begin{aligned}
f_3(N,M) &= -\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}\frac{s_0}{(1-s_0)^{m-1}}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx \\
&+\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}\frac{1-\bar{s}_0}{\bar{s}_0^{m-1}}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx,
\end{aligned}
\tag{1.8}
$$

$$
f_4(N,M) = \int_{s_0}^{1-\bar{s}_0}s\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{s+1}}dxds. \tag{1.9}
$$

2. Estimations of integral results

Let estimate $|f_1(N,M)|$,$|f_2(N,M)|$,$|f_3(N,M)|$ and $|f_4(N,M)|$ for a given complex number s_0 with $|s_0|>1$ and a given small plus value ϵ.

2.1. Estimation of Stirling's formula. Based upon the Stirling's formula

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{\theta}{12n}}, 0<\theta<1,$$

let define a function $\Phi(n,\alpha)=\frac{n!}{\alpha^n}$ which satisfies

$$\Phi(n,\alpha) = \frac{n!}{\alpha^n} = \sqrt{2\pi n}(\frac{n}{e\alpha})^n e^{\frac{\theta}{12n}}, 0<\theta<1 \tag{2.1}$$

and

$$\sqrt{2\pi n}(\frac{n}{e\alpha})^n \le \Phi(n,\alpha) < \sqrt{2\pi n}(\frac{n}{e\alpha})^n e. \tag{2.2}$$

Therefore, we obtain

$$\Phi(M,\alpha) = \frac{M!}{\alpha^M} < \frac{\sqrt{2\pi M}}{e^{M-1}} \texttt{ for } M \le \alpha,$$

$$\Phi(M,\alpha) = \frac{M!}{\alpha^M} < \frac{\sqrt{2\pi M}}{e^{cM-1}} \texttt{ for } M < e\alpha$$

where $c=\log\frac{e\alpha}{M}>0$ and

$$\Phi(M,\alpha) = \frac{M!}{\alpha^M} \ge \sqrt{2\pi M} \texttt{ for } M \ge e\alpha.$$

Thus, when $M\to\infty$, we have

$$\lim_{M\le\alpha\to\infty} \Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}}{e^{M-1}} = 0, \tag{2.3}$$

$$\lim_{M\le\alpha\to\infty} M\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}M}{e^{M-1}} = 0, \tag{2.4}$$

and for $\log N < M, b = 1-\frac{\log N}{M}>0$,

$$\lim_{M\le\alpha\to\infty} N\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}N}{e^{M-1}} = \lim_{M\to\infty} \frac{\sqrt{2\pi M}}{e^{bM-1}} = 0, \tag{2.5}$$

$$\lim_{M\le\alpha\to\infty} NM\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}MN}{e^{M-1}} = \lim_{M\to\infty} \frac{\sqrt{2\pi M}M}{e^{bM-1}} = 0 \tag{2.6}$$

and for $c=\log\frac{e\alpha}{M}>0$

$$\lim_{M<e\alpha\to\infty} \Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}}{e^{cM-1}} = 0, \tag{2.7}$$

$$\lim_{M<e\alpha\to\infty} M\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}M}{e^{cM-1}} = 0 \tag{2.8}$$

and

$$\lim_{M\ge e\alpha\to\infty} \Phi(M,\alpha) \ge \lim_{M\to\infty} \sqrt{2\pi M} = \infty. \tag{2.9}$$

2.2. **Estimation of $|f_4(N,M)|$.** It is obvious that we can find a number N_4 so that when $N \geq N_4$, there is

$$|f_4(N,M)| = |\int_{s_0}^{1-\bar{s}_0} s \int_N^{\infty} \frac{\frac{1}{2}-\{x\}}{x^{s+1}} dxds| \leq \frac{\epsilon}{4}.$$

2.3. **Estimation of $|f_2(N,M)|$.** Since there is

$$|f_2(N,M)| = |\frac{1}{2}\sum_{m=2}^{M} \frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{N^{\sigma_0}} - \frac{1}{\bar{s}_0^{m-1}}\frac{1}{N^{1-\sigma_0}}]N^{-it_0}|$$

$$\leq \frac{1}{2}\sum_{m=2}^{M} \frac{(m-1)!}{\log^m N}[\frac{1}{|1-s_0|^{m-1}}\frac{1}{N^{\sigma_0}} + \frac{1}{|s_0|^{m-1}}\frac{1}{N^{1-\sigma_0}}]$$

$$\leq \sum_{m=2}^{M} \frac{(m-1)!}{\log^m N}\frac{1}{|s_0|^{m-1}N^{\sigma_0}} = \frac{|s_0|}{N^{\sigma_0}}\sum_{m=2}^{M}\frac{\Phi(m,|s_0|\log N)}{m},$$

we can find a number N_2 so that when $N \geq N_2$ and $M < e|s_0|\log N$, there is

$$|f_2(N,M)| \leq \frac{|s_0|}{N^{\sigma_0}}\sum_{m=2}^{M}\frac{\Phi(m,|s_0|\log N)}{m} \leq \frac{|s_0|}{N^{\sigma_0}} M\Phi(M,|s_0|\log N) \leq \frac{\epsilon}{4}.$$

2.4. **Estimation of $|f_3(N,M)|$.** Since there is

$$|f_3(N,M)| = |\sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}\frac{s_0}{(1-s_0)^{m-1}}\int_N^{\infty}\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx$$

$$-\sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}\frac{1-\bar{s}_0}{\bar{s}_0^{m-1}}\int_N^{\infty}\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx|$$

$$\leq \sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}[|\frac{s_0}{(1-s_0)^{m-1}}|\int_N^{\infty}|\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}|dx + |\frac{1-\bar{s}_0}{\bar{s}_0^{m-1}}|\int_N^{\infty}|\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}|dx]$$

$$\leq 2\sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}\frac{|1-\bar{s}_0|}{|s_0|^{m-1}}\int_N^{\infty}|\frac{\frac{1}{2}-\{x\}}{x^{1+\bar{s}_0}}|dx,$$

we can find a number N_3 so that when $N \geq N_3$ and $M < e|s_0|\log N$, there is

$$|f_3(N,M)| \leq 2\sum_{m=1}^{M} \frac{(m-1)!}{\log^m N}\frac{|1-\bar{s}_0|}{|s_0|^{m-1}}\int_N^{\infty}|\frac{\frac{1}{2}-\{x\}}{x^{1+\bar{s}_0}}|dx$$

$$\leq 2|s_0||1-\bar{s}_0|\sum_{m=1}^{M}\frac{\Phi(m,|s_0|\log N)}{m}\int_N^{\infty}|\frac{\frac{1}{2}-\{x\}}{x^{1+\bar{s}_0}}|dx$$

$$\leq 2|s_0||1-\bar{s}_0|M\Phi(M,|s_0|\log N)\int_N^{\infty}|\frac{\frac{1}{2}-\{x\}}{x^{1+\bar{s}_0}}|dx \leq \frac{\epsilon}{4}.$$

2.5. **Estimation of $|f_1(N,M)|$.**

Lemma 2.1. *For $s_0 \neq 1-\bar{s}_0$ and $M \geq e|1-s_0|\log N$, a function*

$$g(N,M)=\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds$$

$$+\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}]$$

is not equal to zero except at most a number M so that there is

$$|\int_{s_0}^{1-\bar{s}_0}(\frac{|1-s_0|}{1-s})^M\frac{N^{1-s}}{1-s}ds|$$

$$\neq|\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{m}\frac{\Phi(m,|1-s_0|\log N)}{\Phi(M,|1-s_0|\log N)}\frac{|1-s_0|^m}{(1-s_0)^{m-1}}[\frac{1}{n^{s_0}}-(\frac{1-s_0}{\bar{s}_0})^{m-1}\frac{1}{n^{1-\bar{s}_0}}]|.$$

Proof. By Equation (1.5), we have

$$g(N,M)=\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{1-s}ds+\frac{1}{2}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{N^{-s_0}}{(1-s_0)^{m-1}}-\frac{N^{-(1-\bar{s}_0)}}{\bar{s}_0^{m-1}}]$$

$$-\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{s_0}{(1-s_0)^{m-1}}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx-\frac{1-\bar{s}_0}{\bar{s}_0^{m-1}}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx]$$

and

$$g(N,M+1)-g(N,M)=\frac{1}{2}\frac{M!}{\log^{M+1}N}[\frac{N^{-s_0}}{(1-s_0)^M}-\frac{N^{-(1-\bar{s}_0)}}{\bar{s}_0^M}]$$

$$-\frac{M!}{\log^{M+1}N}[\frac{s_0}{(1-s_0)^M}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx-\frac{1-\bar{s}_0}{\bar{s}_0^M}\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx].$$

The last equation yields

$$\frac{|g(N,M+1)-g(N,M)|}{\Phi(M,|1-s_0|\log N)}\log N=|\frac{1}{2}[N^{-s_0}-(\frac{1-s_0}{\bar{s}_0})^M N^{-(1-\bar{s}_0)}]$$

$$-[s_0\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx-(1-\bar{s}_0)(\frac{1-s_0}{\bar{s}_0})^M\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx]|$$

$$=|(\frac{1-s_0}{\bar{s}_0})^M[\frac{1}{2}N^{-(1-\bar{s}_0)}-(1-\bar{s}_0)\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}_0}}dx]$$

$$-[\frac{1}{2}N^{-s_0}-s_0\int_N^\infty\frac{\frac{1}{2}-\{x\}}{x^{1+s_0}}dx]|$$

$$=|(\frac{1-s_0}{\bar{s}_0})^M(1-\bar{s}_0)\int_N^\infty\frac{\{x\}}{x^{2-\bar{s}_0}}dx-s_0\int_N^\infty\frac{\{x\}}{x^{1+s_0}}dx|$$

$$=|\int_N^\infty[(\frac{1-s_0}{\bar{s}_0})^M\frac{1-\bar{s}_0}{x^{1-\bar{s}_0}}-\frac{s_0}{x^{s_0}}]\frac{\{x\}}{x}dx|\neq 0$$

where since x is a real variable

$$(\frac{1-s_0}{\bar{s}_0})^M\frac{1-\bar{s}_0}{x^{1-\bar{s}_0}}-\frac{s_0}{x^{s_0}}\neq 0 \texttt{ for } x\geq N.$$

Similarly, for $r=1,2,\cdots$, we have

$$|g(N,M+r)-g(N,M)| = |\sum_{i=1}^{r}[g(N,M+i)-g(N,M+i-1)]|$$

$$= |\int_N^\infty \{\sum_{i=1}^{r}\frac{\Phi(M+i,|1-s_0|\log N)}{\log N}[(\frac{1-s_0}{\bar{s}_0})^{M+i}\frac{1-\bar{s}_0}{x^{1-\bar{s}_0}}-\frac{s_0}{x^{s_0}}]\}\frac{\{x\}}{x}dx| \neq 0$$

where since x is a real variable

$$\sum_{i=1}^{r}\frac{\Phi(M+i,|1-s_0|\log N)}{\log N}[(\frac{1-s_0}{\bar{s}_0})^{M+i}\frac{1-\bar{s}_0}{x^{1-\bar{s}_0}}-\frac{s_0}{x^{s_0}}] \neq 0 \texttt{ for } x \geq N.$$

Thus, for any big positive numbers N and $M \geq e|1-s_0|\log N$, we obtain

$$g(N,M+r) \neq g(N,M) \texttt{ for } r=1,2,\cdots.$$

The inequality means $g(N,M)\neq 0$ except at most a number M, then we have

$$|\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds|$$

$$\neq |\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}]|.$$

The last inequality yields

$$|\int_{s_0}^{1-\bar{s}_0}(\frac{|1-s_0|}{1-s})^M\frac{N^{1-s}}{1-s}ds|$$

$$\neq |\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{m}\frac{\Phi(m,|1-s_0|\log N)}{\Phi(M,|1-s_0|\log N)}\frac{|1-s_0|^m}{(1-s_0)^{m-1}}[\frac{1}{n^{s_0}}-(\frac{1-s_0}{\bar{s}_0})^{m-1}\frac{1}{n^{1-\bar{s}_0}}]|.$$

The proof of the lemma is completed. □

Now, let estimate $|f_1(N,M)|$. As defined by Formula (1.6), there is

$$|f_1(N,M)| = |\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}-\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds$$

$$-\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}]|.$$

By equations (1.6) and (2.3) for $\log N < M \leq |s_0|\log N \to \infty$, we have

$$|f_1(N,M)| \geq |\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}| - |\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds$$

$$+\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}]| = |\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}| \geq 0$$

where when $s_0 \neq 1-\bar{s}_0$, there are

$$|\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}| > 0$$

and

$$|\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds+\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}]|$$

$$\leq \Phi(M,|s_0|\log N)\int_{\sigma_0}^{1-\sigma_0}|\frac{s_0}{1-s}|^M\frac{N^{1-\sigma}}{|1-s|}d\sigma+2\sum_{n=1}^{N}\frac{|s_0|}{n^{\sigma_0}}\sum_{m=1}^{M}\frac{\Phi(m,|s_0|\log N)}{m}$$

$$\leq \Phi(M,|s_0|\log N)N^{1-\sigma_0}[\int_{\sigma_0}^{1-\sigma_0}|\frac{s_0}{1-s}|^M\frac{N^{\sigma_0-\sigma}}{|1-s|}d\sigma+\frac{2|s_0|M}{1-\sigma_0}]=0$$

where by equations (2.4-2.6) and for $\log N < M \leq |s_0|\log N \to \infty$

$$\lim_{M\to\infty}N^{1-\sigma_0}\Phi(M,|s_0|\log N)=0$$

and

$$\lim_{M\to\infty}N^{1-\sigma_0}M\Phi(M,|s_0|\log N)=0.$$

On the other hand, by Equation (2.9) for $M \geq e|1-s_0|\log N \to \infty$, we have

$$|f_1(N,M)|\geq|\frac{M!}{\log^M N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+1}}ds$$

$$+\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(m-1)!}{\log^m N}[\frac{1}{(1-s_0)^{m-1}}\frac{1}{n^{s_0}}-\frac{1}{\bar{s}_0^{m-1}}\frac{1}{n^{1-\bar{s}_0}}]|-|\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}|$$

$$\geq \Phi(M,|1-s_0|\log N)N^{1-\sigma_0}\{|\int_{s_0}^{1-\bar{s}_0}(\frac{|1-s_0|}{1-s})^M\frac{N^{\sigma_0-s}}{1-s}ds$$

$$+\frac{|1-s_0|}{N^{1-\sigma_0}}\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{m}\frac{\Phi(m,|1-s_0|\log N)}{\Phi(M,|1-s_0|\log N)}(\frac{|1-s_0|}{1-s_0})^{m-1}[\frac{1}{n^{s_0}}-(\frac{1-s_0}{\bar{s}_0})^{m-1}\frac{1}{n^{1-\bar{s}_0}}]|$$

$$-\frac{1}{\Phi(M,|1-s_0|\log N)N^{1-\sigma_0}}|\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}|\}$$

$$=\Phi(M,|1-s_0|\log N)N^{1-\sigma_0}|\int_{s_0}^{1-\bar{s}_0}(\frac{|1-s_0|}{1-s})^M\frac{N^{1-s}}{1-s}ds$$

$$+\frac{|1-s_0|}{N^{1-\sigma_0}}\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{m}\frac{\Phi(m,|1-s_0|\log N)}{\Phi(M,|1-s_0|\log N)}(\frac{|1-s_0|}{1-s_0})^{m-1}[\frac{1}{n^{s_0}}-(\frac{1-s_0}{\bar{s}_0})^{m-1}\frac{1}{n^{1-\bar{s}_0}}]|\geq 0$$

where by Lemma (2.1) for $s_0 \neq 1-\bar{s}_0$

$$|\int_{s_0}^{1-\bar{s}_0}(\frac{|1-s_0|}{1-s})^M\frac{N^{\sigma_0-s}}{1-s}ds$$

$$+\frac{|1-s_0|}{N^{1-\sigma_0}}\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{1}{m}\frac{\Phi(m,|1-s_0|\log N)}{\Phi(M,|1-s_0|\log N)}(\frac{|1-s_0|}{1-s_0})^{m-1}[\frac{1}{n^{s_0}}-(\frac{1-s_0}{\bar{s}_0})^{m-1}\frac{1}{n^{1-\bar{s}_0}}]|>0$$

and by equations (1.4) and (2.9)

$$\frac{N^{-(1-\sigma_0)}}{\Phi(M,|1-s_0|\log N)}|\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}|\leq\frac{N^{-(1-\sigma_0)}}{\Phi(M,|1-s_0|\log N)}\frac{N^{1-\sigma_0}-N^{\sigma_0}}{\sigma_0\log N}=0.$$

Thus, for $\log N < M \le |s_0| \log N \to \infty$ we have

$$|\sum_{n=1}^{N} \int_{s_0}^{1-\bar{s}_0} \frac{ds}{n^s}| \ge |\frac{M!}{\log^M N} \int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+1}} ds$$
$$+ \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{(m-1)!}{\log^m N} [\frac{1}{(1-s_0)^{m-1}} \frac{1}{n^{s_0}} - \frac{1}{\bar{s}_0^{m-1}} \frac{1}{n^{1-\bar{s}_0}}]|$$

and for $M \ge e|1-s_0| \log N \to \infty$ we have

$$|\sum_{n=1}^{N} \int_{s_0}^{1-\bar{s}_0} \frac{ds}{n^s}| \le |\frac{M!}{\log^M N} \int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+1}} ds$$
$$+ \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{(m-1)!}{\log^m N} [\frac{1}{(1-s_0)^{m-1}} \frac{1}{n^{s_0}} - \frac{1}{\bar{s}_0^{m-1}} \frac{1}{n^{1-\bar{s}_0}}]|.$$

Therefore, when $|s_0| \log N < M < e|1-s_0| \log N$, some numbers M can satisfy

$$-\frac{\epsilon}{4} \le |\sum_{n=1}^{N} \int_{s_0}^{1-\bar{s}_0} \frac{ds}{n^s}| - |\frac{M!}{\log^M N} \int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+1}} ds$$
$$+ \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{(m-1)!}{\log^m N} [\frac{1}{(1-s_0)^{m-1}} \frac{1}{n^{s_0}} - \frac{1}{\bar{s}_0^{m-1}} \frac{1}{n^{1-\bar{s}_0}}]| \le \frac{\epsilon}{4}.$$

Furthermore, since $\lim_{N\to\infty} f_1(N, M)$ should exist, we can find a number N_1 so that when $N \ge N_1$ and $|s_0| \log N < M < e|1-s_0| \log N$, there is

$$|f_1(N, M)| = |\sum_{n=1}^{N} \int_{s_0}^{1-\bar{s}_0} \frac{ds}{n^s} - \frac{M!}{\log^M N} \int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+1}} ds$$
$$- \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{(m-1)!}{\log^m N} [\frac{1}{(1-s_0)^{m-1}} \frac{1}{n^{s_0}} - \frac{1}{\bar{s}_0^{m-1}} \frac{1}{n^{1-\bar{s}_0}}]| \le \frac{\epsilon}{4}.$$

3. Lemma of central dissymmetry of $\zeta(s)$

Lemma 3.1. *The Riemann zeta function $\zeta(s) = u(\sigma, t) + iv(\sigma, t)$ is central unsymmetrical for the variable σ in the open region $(\sigma_0, 1-\sigma_0)$ about the point $(\sigma = \frac{1}{2}, t_0)$ along a horizontal line $t = t_0$ from s_0 to $1 - \bar{s}_0$ except for the zeros of $\zeta(s) = 0$. This means that the following two equations*

$$u(\sigma, t_0) + u(1-\sigma, t_0) \equiv 0 \text{ and } v(\sigma, t_0) + v(1-\sigma, t_0) \equiv 0 \tag{3.1}$$

do not hold.

Proof. Based upon Expression (0.1), let write

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - \frac{1}{2} N^{-s} + s \int_N^{\infty} \frac{\frac{1}{2} - \{x\}}{x^{s+1}} dx = u(\sigma, t) + iv(\sigma, t)$$

where

$$u(\sigma, t) = \sum_{n=1}^{N} \frac{1}{n^\sigma} cos(t \log n)$$
$$-[(1-\sigma) cos(t \log N) + t sin(t \log N)] \frac{N^{1-\sigma}}{|1-s|^2} - \frac{1}{2} N^{-\sigma} cos(t \log N)$$

$$+\sigma\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{1+\sigma}}cos(t\log x)dx+t\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{1+\sigma}}sin(t\log x)dx$$

$$v(\sigma,t)=-\sum_{n=1}^{N}\frac{1}{n^\sigma}sin(t\log n)$$

$$+[(1-\sigma)sin(t\log N)-tcos(t\log N)]\frac{N^{1-\sigma}}{|1-s|^2}+\frac{1}{2}N^{-\sigma}sin(t\log N)$$

$$+t\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{1+\sigma}}cos(t\log x)dx-\sigma\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{1+\sigma}}sin(t\log x)dx$$

and

$$\zeta(1-\bar{s})=\sum_{n=1}^{N}\frac{1}{n^{1-\bar{s}}}-\frac{N^{\bar{s}}}{\bar{s}}-\frac{1}{2}N^{-(1-\bar{s})}+(1-\bar{s})\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{2-\bar{s}}}dx$$

$$=u(1-\sigma,t)+iv(1-\sigma,t).$$

where

$$u(1-\sigma,t)=\sum_{n=1}^{N}\frac{1}{n^{1-\sigma}}cos(t\log n)$$

$$-[\sigma cos(t\log N)+tsin(t\log N)]\frac{N^\sigma}{|1-s|^2}-\frac{1}{2}N^{-(1-\sigma)}cos(t\log N)$$

$$+(1-\sigma)\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{2-\sigma}}cos(t\log x)dx+t\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{2-\sigma}}sin(t\log x)dx$$

$$v(1-\sigma,t)=-\sum_{n=1}^{N}\frac{1}{n^{1-\sigma}}sin(t\log n)$$

$$+[\sigma sin(t\log N)-tcos(t\log N)]\frac{N^\sigma}{|1-s|^2}+\frac{1}{2}N^{-(1-\sigma)}sin(t\log N)$$

$$+t\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{2-\sigma}}cos(t\log x)dx-(1-\sigma)\int_N^\infty \frac{\frac{1}{2}-\{x\}}{x^{2-\sigma}}sin(t\log x)dx$$

By comparing $u(\sigma,t)$ with $-u(1-\sigma,t)$ and $v(\sigma,t)$ with $-v(1-\sigma,t)$, it is obvious that the following two equations

$$u(\sigma,t)\equiv -u(1-\sigma,t)$$

and

$$v(\sigma,t)\equiv -v(1-\sigma,t)$$

do not hold or the following two equations

$$u(\sigma,t)+u(1-\sigma,t)\equiv 0$$

and

$$v(\sigma,t)+v(1-\sigma,t)\equiv 0$$

do not hold.

The proof of the lemma is completed. □

4. Proof of the Riemann hypothesis

Theorem 4.1. *The nontrivial zeros of the Riemann zeta function $\zeta(s)$ all have real part $\Re(s) = \frac{1}{2}$.*

Proof. Based upon the analysis and estimations of the curve integral (1.3) in above sections, for a given small value ϵ, we can find a number N_ϵ so that when $N \geq N_\epsilon$ and $|s_0| \log N < M < e|s_0| \log N$ and $|s_0| > 1$, there is

$$|F(1-\bar{s}_0) - F(s_0)| = |\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds|$$

$$\leq |f_1(N,M)| + |f_2(N,M)| + |f_3(N,M)| + |f_4(N,M)| \leq \epsilon.$$

Because ϵ is arbitrarily small, we can get

$$\lim_{\epsilon \to 0} |F(1-\bar{s}_0) - F(s_0)| = \lim_{\epsilon \to 0} |\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds| = 0.$$

Since the integral $|\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds|$ is a determined value and independent of the number M or the value $\frac{M}{\log N}$ for any integer $N \geq 1$, thus, the curve integral (1.3) becomes

$$|\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds| = |\int_{\sigma_0}^{1-\sigma_0} [u(\sigma,t_0) + iv(\sigma,t_0)] d\sigma| = 0 \tag{4.1}$$

for $|s_0| > 1$, where

$$\zeta(s) = u(\sigma,t) + iv(\sigma,t).$$

Since the curve integral is only dependent of the start and end points of the curve, therefore, at least one of some equations has to be satisfied. These equations are

$$s_0 = 1 - \bar{s}_0$$

which means that σ_0 can be determined by $s_0 = 1 - \bar{s}_0$ since

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds = \int_{s_0}^{s_0} \zeta(s) ds = 0,$$

$$u(\sigma,t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds = i \int_{\sigma_0}^{1-\sigma_0} v(\sigma,t_0) d\sigma = i f_v(\sigma_0) = 0,$$

$$v(\sigma,t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds = \int_{\sigma_0}^{1-\sigma_0} u(\sigma,t_0) d\sigma = f_u(\sigma_0) = 0,$$

$$u(\sigma,t_0) = v(\sigma,t_0)$$

which means that σ_0 can be determined by

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s) ds = (1+i) \int_{\sigma_0}^{1-\sigma_0} u(\sigma,t_0) d\sigma = (1+i) f_u(\sigma_0) = 0,$$

$$u(\sigma,t_0) + u(1-\sigma,t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{\sigma_0}^{1-\sigma_0} u(\sigma,t_0)d\sigma = \int_{\sigma_0}^{1/2}[u(\sigma,t_0)+u(1-\sigma,t_0)]d\sigma \equiv 0$$

and

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = i\int_{\sigma_0}^{1-\sigma_0} v(\sigma,t_0)d\sigma = if_v(\sigma_0) = 0,$$

and

$$v(\sigma,t_0)+v(1-\sigma,t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{\sigma_0}^{1-\sigma_0} v(\sigma,t_0)d\sigma = \int_{\sigma_0}^{1/2}[v(\sigma,t_0)+v(1-\sigma,t_0)]d\sigma \equiv 0$$

and

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = \int_{\sigma_0}^{1-\sigma_0} u(\sigma,t_0)d\sigma = f_u(\sigma_0) = 0$$

along a horizontal line $t = t_0$ from s_0 to $1-\bar{s}_0$ because of the continuity of the Riemann zeta function $\zeta(s)$, which means that $\zeta(s)$ must be central symmetric about the point $\sigma = \frac{1}{2}$ on the horizontal line.

First, $u(\sigma,t_0)$ and $v(\sigma,t_0)$ are not always equal to zero along a horizontal line from s to $1-\bar{s}$.

Next, it is obvious that there is

$$u(\sigma,t_0) \neq v(\sigma,t_0).$$

Thirdly, by Lemma (3.1), the following two equations

$$u(\sigma,t_0)+u(1-\sigma,t_0) \equiv 0$$

and

$$v(\sigma,t_0)+v(1-\sigma,t_0) \equiv 0$$

do not hold for a fixed value $t = t_0$. Thus, there is no value of σ_0 in the region $0 < \sigma_0 < 1$ except for $\sigma_0 = \frac{1}{2}$, which satisfies a group of two different equations

$$f_u(\sigma_0) = \int_{\sigma_0}^{1-\sigma_0} u(\sigma,t_0)d\sigma = 0$$

and

$$f_v(\sigma_0) = \int_{\sigma_0}^{1-\sigma_0} v(\sigma,t_0)d\sigma = 0.$$

When equations $u(\sigma,t_0) \equiv 0$, $v(\sigma,t_0) \equiv 0$, $u(\sigma,t_0) = v(\sigma,t_0)$, $u(\sigma,t_0)+u(1-\sigma,t_0) = 0$ and $v(\sigma,t_0)+v(1-\sigma,t_0) = 0$ are not satisfied, the remained equation

$$s_0 = 1-\bar{s}_0$$

must be satisfied, that is, there must be

$$\sigma_0 + it_0 = 1-\sigma_0+it_0.$$

By this equation, we get

$$\sigma_0 = \frac{1}{2}.$$

Since s_0 is any nontrivial zero of the Riemann zeta function $\zeta(s)$, there must be

$$\Re(s) = \frac{1}{2}$$

for any nontrivial zero of the Riemann zeta function $\zeta(s)$ and for $|s_0| > 1$.

For the limited region $|s_0| \leq 1$, where s in the region $\{\sigma_0 \leq \sigma \leq 1-\sigma_0, t = t_0\}$, we can calculate nontrivial zeros of the Riemann zeta function $\zeta(s)$ to verify whether they have real part $\Re(s) = \frac{1}{2}$. Actually, computations of nontrivial zeros of $\zeta(s)$ have been made in a region $[-T, T]$ for their imaginary parts, where $T \cong 5 \times 10^8$. Results of these computations show that nontrivial zeros of the Riemann zeta function $\zeta(s)$ in the region $[-T, T]$ have real part $\Re(s) = \frac{1}{2}$. The Riemann hypothesis was computationally tested and found to be true for the first zeros by Brent et al. (1982), covering zeros in the region $0 < t < 81702130.19$. A computation made by Gourdon (2004) verifies that the Riemann hypothesis is true at least for all less than 2.4 trillion.

This completes the proof of the theorem and the Riemann hypothesis is proved. $\square$

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Part 2. On the distribution of nontrivial zeros of the Riemann zeta function by using the integral representation of $\zeta(s)$

5. INTRODUCTION

It is well known that the Riemann zeta function $\zeta(s)$ of a complex variable $s = \sigma + it$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for the real part $\Re(s) > 1$ and its analytic continuation in the critical strip $0 < \sigma < 1$ is proved as following (See representation (2.1.5) on page 14 of the reference book[1])

$$\zeta(s) = s \int_0^{\infty} \frac{[x]-x}{x^{s+1}} dx, 0 < \sigma < 1. \tag{5.1}$$

It extends to an analytic function in the whole complex plane except for having a simple pole at $s = 1$. Trivially, $\zeta(-2n) = 0$ for all positive integers. All other zeros of the Riemann zeta functions are called its nontrivial zeros[1-6].

The Riemann hypothesis states that nontrivial zeros of $\zeta(s)$ all have real part $\Re(s) = \frac{1}{2}$.

The investigation of the distribution of nontrivial zeros of the Riemann zeta function in this paper includes several parts. First, a curve integral of the Riemann zeta function $\zeta(s)$ was formed, which is along a horizontal line from s to $1-\bar{s}$ which are two nontrivial zeros of $\zeta(s)$ and symmetric about the vertical line $\sigma = \frac{1}{2}$. Next, the result of the curve integral was derived and proved equal to zero. Then, by proving a lemma of central dissymmetry of the Riemann zeta function $\zeta(s)$, two nontrivial zeros s and $1-\bar{s}$ were proved being a same zero or satisfying $s = 1-\bar{s}$. Hence, nontrivial zeros of $\zeta(s)$ all have real part $\Re(s) = \frac{1}{2}$.

The distribution of nontrivial zeros of the Riemann zeta function was also investigated in the author's another paper by using another representation of Formula (5.1)[1-3]

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - \frac{1}{2}N^{-s} + s\int_N^\infty \frac{\frac{1}{2} - \{x\}}{x^{s+1}}dx \tag{5.2}$$

for any integer $N \geq 1$ and $\Re(s) > 0$.

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6. Curve integral

The Riemann zeta function $\zeta(s)$ defined in Formula (5.1) is analytic in a simply connected complex region D. According to the Cauchy theorem, suppose that C is any simple closed curve in D, then, there is

$$\oint_C \zeta(s)ds = 0. \tag{6.1}$$

According to the theory of the Riemann zeta function[1-3], the nontrivial zeros of $\zeta(s)$ are symmetric about the vertical line $\sigma = \frac{1}{2}$. When a complex number $s_0 = \sigma_0 + it_0$ is a nontrivial zero of $\zeta(s)$, the complex number $1 - \sigma_0 + it_0 = 1 - \bar{s}_0$ is also a nontrivial zero of $\zeta(s)$, and there must be $\zeta(s_0) = 0$ and $\zeta(1-\bar{s}_0) = 0$.

Since $\zeta(s_0) = 0$ and $\zeta(1-\bar{s}_0) = 0$, by using Formula (5.1) and denoting

$$\eta(s) = \int_0^\infty \frac{[x]-x}{x^{s+1}}dx,$$

there are

$$\eta'(s) = -\int_0^\infty \frac{[x]-x}{x^{s+1}}\log x dx,$$

$$\zeta(s_0) = s_0\int_0^\infty \frac{[x]-x}{x^{1+s_0}}dx = s_0\eta(s_0) = 0 \tag{6.2}$$

and

$$\zeta(1-\bar{s}_0) = (1-\bar{s}_0)\int_0^\infty \frac{[x]-x}{x^{2-\bar{s}_0}}dx = (1-\bar{s}_0)\eta(1-\bar{s}_0) = 0. \tag{6.3}$$

Let consider the curve integral of $\zeta(s)$ along a horizontal line $t = t_0$ in the region $0 < \sigma < 1$ from s_0 to $1 - \bar{s}_0$ as following

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = Z(1-\bar{s}_0) - Z(s_0) \ \texttt{for} \ 0 < \sigma_0 \leq \frac{1}{2} \tag{6.4}$$

where $Z(s)$ is the primitive function of $\zeta(s)$.

By using Formula (5.1), Equation (6.4) can be written as

$$Z(1-\bar{s}_0) - Z(s_0) = \int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = \int_{s_0}^{1-\bar{s}_0} s \int_0^\infty \frac{[x]-x}{x^{s+1}} dxds. \tag{6.5}$$

The integral on the right hand side of Equation (6.5) is

$$\int_{s_0}^{1-\bar{s}_0} s \int_0^\infty \frac{[x]-x}{x^{s+1}} dxds = \int_{s_0}^{1-\bar{s}_0} s\eta(s)ds$$

$$= \frac{s^2}{2}\eta(s)|_{s_0}^{1-\bar{s}_0} - \int_{s_0}^{1-\bar{s}_0} \frac{s^2}{2}\eta'(s)ds = \frac{1}{2}\int_{s_0}^{1-\bar{s}_0} s^2 \int_0^\infty \frac{[x]-x}{x^{s+1}} \log x dxds$$

where by using equations (6.2-6.3), there is

$$\frac{s^2}{2}\eta(s)|_{s_0}^{1-\bar{s}_0} = \frac{(1-\bar{s}_0)^2}{2}\eta(1-\bar{s}_0) - \frac{s_0^2}{2}\eta(s_0) = 0.$$

Then, by denoting

$$F'(x) = f(x) = -\frac{1}{(n+x)^{s+1}} \ \texttt{for} \ 0 \leq x \leq 1,$$

$$G(x) = -([n+x]-(n+x))\log(n+x) = x\log(n+x) \ \texttt{for} \ 0 \leq x \leq 1,$$

we have

$$F(x) = \frac{1}{s(n+x)^s}, G'(x) = \frac{x}{n+x} + \log(n+x).$$

Since there is

$$\int_0^\infty \frac{[x]-x}{x^{s+1}} \log x dx = \sum_{n=0}^\infty \int_0^1 f(x)G(x)dx$$

$$= \sum_{n=0}^\infty F(x)G(x)|_0^1 - \sum_{n=0}^\infty \int_0^1 F(x)G'(x)dx,$$

by substituting $F(x)$, $G(x)$ and $G'(x)$ into the equation, we have

$$\int_0^\infty \frac{[x]-x}{x^{s+1}} \log x dx = \sum_{n=0}^\infty \frac{x\log(n+x)}{s(n+x)^s}|_0^1 - \sum_{n=0}^\infty \int_0^1 [\frac{x}{s(n+x)^{s+1}} + \frac{\log(n+x)}{s(n+x)^s}]dx$$

$$= \frac{1}{s}\lim_{N\to\infty}\sum_{n=0}^N \frac{\log(n+1)}{(n+1)^s} + \frac{1}{s}\int_0^\infty \frac{[x]-x}{x^{s+1}}dx - \frac{1}{s}\lim_{N\to\infty}\int_0^N \frac{\log x}{x^s}dx$$

$$= \frac{1}{s}\int_0^\infty \frac{[x]-x}{x^{s+1}}dx + \frac{1}{s}\lim_{N\to\infty}\sum_{n=1}^N \frac{\log n}{n^s} - \frac{1}{s}\lim_{N\to\infty}\int_0^N \frac{\log x}{x^s}dx.$$

Integrating the last integral on the right hand side of the equation, we have

$$\int_0^N \frac{\log x}{x^s}dx = \frac{1}{1-s}(x^{1-s}\log x)|_0^N - \frac{1}{1-s}\int_0^N \frac{1}{x^s}dx$$

$$= \frac{1}{1-s}(x^{1-s}\log x)|_0^N - \frac{1}{(1-s)^2}x^{1-s}|_0^N$$
$$= \frac{1}{1-s}[(\log x - \frac{1}{1-s})x^{1-s}]|_0^N.$$

Therefore, we have

$$\int_{s_0}^{1-\bar{s}_0} s\int_0^\infty \frac{[x]-x}{x^{s+1}}dxds = \frac{1}{2}\int_{s_0}^{1-\bar{s}_0} s^2\int_0^\infty \frac{[x]-x}{x^{s+1}}\log xdxds$$
$$= \frac{1}{2}\int_{s_0}^{1-\bar{s}_0} s\{\int_0^\infty \frac{[x]-x}{x^{s+1}}dx + \lim_{N\to\infty}\sum_{n=1}^{N}\frac{\log n}{n^s}$$
$$-\frac{1}{1-s}\lim_{N\to\infty}[(\log x - \frac{1}{1-s})x^{1-s}]|_0^N\}ds$$
$$= \frac{1}{2}\int_{s_0}^{1-\bar{s}_0} s\int_0^\infty \frac{[x]-x}{x^{s+1}}dxds + \frac{1}{2}\lim_{N\to\infty}\int_{s_0}^{1-\bar{s}_0} s\sum_{n=1}^{N}\frac{\log n}{n^s}ds$$
$$-\frac{1}{2}\lim_{N\to\infty}\int_{s_0}^{1-\bar{s}_0}\frac{s}{1-s}(\log N - \frac{1}{1-s})N^{1-s}]ds.$$

By arranging the equation, we have

$$\int_{s_0}^{1-\bar{s}_0}\zeta(s)ds = \int_{s_0}^{1-\bar{s}_0} s\int_0^\infty \frac{[x]-x}{x^{s+1}}dxds = \lim_{N\to\infty}\Delta(s_0,N) \tag{6.6}$$

where

$$\Delta(s_0,N) = \int_{s_0}^{1-\bar{s}_0} s\sum_{n=1}^{N}\frac{\log n}{n^s}ds - \int_{s_0}^{1-\bar{s}_0}\frac{s}{1-s}(\log N - \frac{1}{1-s})N^{1-s}ds.$$

Since the left hand side of Eq. (6.6) is the integral of an analytic function in the field of its definition and with a finite value, the limit on the right hand side of the equation must exist and equal to the finite value.

By integrating the integrals in $\Delta(s_0,N)$, we have

$$\int_{s_0}^{1-\bar{s}_0} s\sum_{n=1}^{N}\frac{\log n}{n^s}ds = \sum_{n=1}^{N}(\frac{s_0}{n^{s_0}} - \frac{1-\bar{s}_0}{n^{1-\bar{s}_0}}) + \sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}$$

and

$$\int_{s_0}^{1-\bar{s}_0}\frac{s}{1-s}(\log N - \frac{1}{1-s})N^{1-s}ds = -\int_{s_0}^{1-\bar{s}_0}(1-\frac{1}{1-s})N^{1-s}\log Nds$$
$$+\int_{s_0}^{1-\bar{s}_0}[\frac{1}{1-s} - \frac{1}{(1-s)^2}]N^{1-s}ds$$
$$= (1-\frac{1}{1-s})N^{1-s}|_{s_0}^{1-\bar{s}_0} - \frac{1}{\log N}[\frac{1}{1-s} - \frac{1}{(1-s)^2}]N^{1-s}|_{s_0}^{1-\bar{s}_0}$$
$$+\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^2}ds + \frac{1}{\log N}\int_{s_0}^{1-\bar{s}_0}[\frac{1}{(1-s)^2} - \frac{2}{(1-s)^3}]N^{1-s}ds.$$

Since there is

$$\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^2}ds = -\frac{1}{\log N}\frac{1}{(1-s)^2}N^{1-s}|_{s_0}^{1-\bar{s}_0} + \frac{1}{\log N}\int_{s_0}^{1-\bar{s}_0}\frac{2}{(1-s)^3}N^{1-s}ds,$$

by arranging the last equation, we have

$$\int_{s_0}^{1-\bar{s}_0}\frac{s}{1-s}(\log N-\frac{1}{1-s})N^{1-s}ds=[(1-\frac{1}{\bar{s}_0})N^{\bar{s}_0}-(1-\frac{1}{1-s_0})N^{1-s_0}]$$

$$-\frac{1}{\log N}(\frac{N^{\bar{s}_0}}{\bar{s}_0}-\frac{N^{1-s_0}}{1-s_0})+\frac{1}{\log N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^2}ds$$

$$=-(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})-\frac{1}{\log N}(\frac{N^{\bar{s}_0}}{\bar{s}_0}-\frac{N^{1-s_0}}{1-s_0})+\frac{1}{\log N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^2}ds.$$

By integrating the last integral in the above equation, we get

$$\int_{s_0}^{1-\bar{s}_0}\frac{s}{1-s}(\log N-\frac{1}{1-s})N^{1-s}ds=-(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})$$

$$-\sum_{m=0}^{M}\frac{m!}{\log^{m+1}N}[\frac{N^{\bar{s}_0}}{\bar{s}_0^{m+1}}-\frac{N^{1-s_0}}{(1-s_0)^{m+1}}]+\frac{(M+1)!}{\log^{M+1}N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+2}}ds.$$

Hence, we have

$$\Delta(s_0,N)=\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})$$
$$+\sum_{m=0}^{M}\frac{m!}{\log^{m+1}N}[\frac{N^{\bar{s}_0}}{\bar{s}_0^{m+1}}-\frac{N^{1-s_0}}{(1-s_0)^{m+1}}]-\frac{(M+1)!}{\log^{M+1}N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+2}}ds. \tag{6.7}$$

It should be pointed out that although each one of the five terms in $\Delta(s_0,N)$ may tend to ∞ when $N\to\infty$, $\lim_{N\to\infty}\Delta(s_0,N)$ exists and is a finite value. We will estimate the finite value of $\Delta(s_0,N)$ when $N\to\infty$ in next section.

7. Estimation of $\Delta(s_0,N)$ when $N\to\infty$

7.1. **Estimation of Stirling's formula.** Based upon the Stirling's formula

$$n!=\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{\theta}{12n}},0<\theta<1,$$

let define a function $\Phi(n,\alpha)=\frac{n!}{\alpha^n}$ which satisfies

$$\Phi(n,\alpha)=\frac{n!}{\alpha^n}=\sqrt{2\pi n}(\frac{n}{e\alpha})^n e^{\frac{\theta}{12n}},0<\theta<1 \tag{7.1}$$

and

$$\sqrt{2\pi n}(\frac{n}{e\alpha})^n\leq\Phi(n,\alpha)<\sqrt{2\pi n}(\frac{n}{e\alpha})^n e. \tag{7.2}$$

Therefore, we obtain

$$\Phi(M,\alpha)=\frac{M!}{\alpha^M}<\frac{\sqrt{2\pi M}}{e^{M-1}}\ \texttt{for}\ M\leq\alpha,$$

$$\Phi(M,\alpha)=\frac{M!}{\alpha^M}<\frac{\sqrt{2\pi M}}{e^{cM-1}}\ \texttt{for}\ M<e\alpha$$

where $c=\log\frac{e\alpha}{M}>0$ and

$$\Phi(M,\alpha)=\frac{M!}{\alpha^M}\geq\sqrt{2\pi M}\ \texttt{for}\ M\geq e\alpha.$$

Thus, when $M \to \infty$, we have

$$\lim_{M \le \alpha \to \infty} \Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}}{e^{M-1}} = 0, \tag{7.3}$$

$$\lim_{M \le \alpha \to \infty} M\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}M}{e^{M-1}} = 0, \tag{7.4}$$

and for $\log N < M, b = 1 - \frac{\log N}{M} > 0$,

$$\lim_{M \le \alpha \to \infty} N\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}N}{e^{M-1}} = \lim_{M\to\infty} \frac{\sqrt{2\pi M}}{e^{bM-1}} = 0, \tag{7.5}$$

$$\lim_{M \le \alpha \to \infty} NM\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}MN}{e^{M-1}} = \lim_{M\to\infty} \frac{\sqrt{2\pi M}M}{e^{bM-1}} = 0 \tag{7.6}$$

and for $c = \log \frac{e\alpha}{M} > 0$

$$\lim_{M < e\alpha \to \infty} \Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}}{e^{cM-1}} = 0, \tag{7.7}$$

$$\lim_{M < e\alpha \to \infty} M\Phi(M,\alpha) \le \lim_{M\to\infty} \frac{\sqrt{2\pi M}M}{e^{cM-1}} = 0 \tag{7.8}$$

and

$$\lim_{M \ge e\alpha \to \infty} \Phi(M,\alpha) \ge \lim_{M\to\infty} \sqrt{2\pi M} = \infty. \tag{7.9}$$

7.2. **Estimation of sum I.** The first sum on the right hand side of Eq. (6.7) can be estimated as

$$\begin{aligned} |\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}} - \frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})| &\le \sum_{n=1}^{N}[\frac{|s_0|}{n^{\sigma_0}} + \frac{|1-\bar{s}_0|}{n^{1-\sigma_0}}] \\ &\le 2|1-\bar{s}_0|\int_0^N \frac{dx}{x^{\sigma_0}} \le 2|1-\bar{s}_0|\frac{N^{1-\sigma_0}}{1-\sigma_0}. \end{aligned}$$

Thus, we have

$$|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}} - \frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})| \le 2|1-\bar{s}_0|\frac{N^{1-\sigma_0}}{1-\sigma_0}. \tag{7.10}$$

7.3. **Estimation of integral II.** The second integral on the right hand side of Eq. (6.7) can be estimated as follows:

$$\begin{aligned} |\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}| &\le \sum_{n=1}^{N}\int_{\sigma_0}^{1-\sigma_0}\frac{d\sigma}{n^\sigma} = \int_{\sigma_0}^{1-\sigma_0}\sum_{n=1}^{N}\frac{d\sigma}{n^\sigma} \le \int_{\sigma_0}^{1-\sigma_0}\int_0^N \frac{dx}{x^\sigma}d\sigma \\ &= \int_{\sigma_0}^{1-\sigma_0}\frac{N^{1-\sigma}}{1-\sigma}d\sigma \le \int_{\sigma_0}^{1-\sigma_0}\frac{N^{1-\sigma}}{\sigma_0}d\sigma = \frac{N^{1-\sigma_0}-N^{\sigma_0}}{\sigma_0\log N}. \end{aligned}$$

Thus, we have

$$|\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{1}{n^s}ds| \le \frac{N^{1-\sigma_0}-N^{\sigma_0}}{\sigma_0\log N}. \tag{7.11}$$

7.4. **Estimation of $\Delta(s_0,N)$.** For $|s_0|>1$ and by equations (6.7) and (7.3-7.6) for $\log N\le M+1\le|s_0|\log N\to\infty$, we have

$$|\Delta(s_0,N)|\ge|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})|$$

$$-|\sum_{m=0}^{M}\frac{m!}{\log^{m+1}N}[\frac{N^{\bar{s}_0}}{\bar{s}_0^{m+1}}-\frac{N^{1-s_0}}{(1-s_0)^{m+1}}]-\frac{(M+1)!}{\log^{M+1}N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+2}}ds|$$

$$\ge N^{1-\sigma_0}\{|N^{s_0-1}[\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}]+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{2\sigma_0-1}-\frac{s_0}{1-s_0})|$$

$$-|\sum_{m=0}^{M}\frac{\Phi(m+1,|s_0|\log N)}{m+1}[N^{2\sigma_0-1}-(\frac{\bar{s}_0}{1-s_0})^{m+1}]$$

$$-\Phi(M+1,|s_0|\log N)\int_{s_0}^{1-\bar{s}_0}(\frac{|s_0|}{1-s})^{M+1}\frac{N^{s_0-s}}{1-s}ds|\}$$

$$=|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})|\ge 0$$

where when $s_0\ne 1-\bar{s}_0$, there are

$$|N^{s_0-1}[\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}]+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{2\sigma_0-1}-\frac{s_0}{1-s_0})|>0$$

and

$$|\sum_{m=0}^{M}\frac{\Phi(m+1,|s_0|\log N)}{m+1}[N^{2\sigma_0-1}-(\frac{\bar{s}_0}{1-s_0})^{m+1}]$$

$$-\Phi(M+1,|s_0|\log N)\int_{s_0}^{1-\bar{s}_0}(\frac{|s_0|}{1-s})^{M+1}\frac{N^{s_0-s}}{1-s}ds|$$

$$\le\sum_{m=0}^{M}\frac{\Phi(m+1,|s_0|\log N)}{m+1}[N^{2\sigma_0-1}+|\frac{\bar{s}_0}{1-s_0}|^{m+1}]$$

$$+\Phi(M+1,|s_0|\log N)|\int_{s_0}^{1-\bar{s}_0}(\frac{|s_0|}{1-s})^{M+1}\frac{N^{\sigma_0-s}}{1-s}ds|$$

$$\le\Phi(M+1,|s_0|\log N)[M+|\int_{s_0}^{1-\bar{s}_0}(\frac{|s_0|}{1-s})^{M+1}\frac{N^{s_0-s}}{1-s}ds|]=0$$

where by equations (7.3-7.6)

$$\lim_{M\to\infty}\Phi(M+1,|s_0|\log N)=0$$

and

$$\lim_{M\to\infty}M\Phi(M+1,|s_0|\log N)=0.$$

On the other hand, for $|s_0|>1$ and by equations (6.7) and (7.9) for $M+1\ge e|1-s_0|\log N\to\infty$, we have

$$|\Delta(s_0,N)|\ge|\frac{(M+1)!}{\log^{M+1}N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+2}}ds-\sum_{m=0}^{M}\frac{m!}{\log^{m+1}N}[\frac{N^{\bar{s}_0}}{\bar{s}_0^{m+1}}-\frac{N^{1-s_0}}{(1-s_0)^{m+1}}]|$$

$$-|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})|$$

$$\geq \Phi(M+1,|1-s_0|\log N)N^{1-\sigma_0}\{|\int_{\sigma_0}^{1-\sigma_0}(\frac{|1-s_0|}{1-s})^{M+1}\frac{N^{\sigma_0-\sigma}}{1-s}d\sigma$$

$$-\sum_{m=0}^{M}\frac{1}{m+1}\frac{\Phi(m+1,|1-s_0|\log N)}{\Phi(M+1,|1-s_0|\log N)}(\frac{|1-s_0|}{1-s_0})^{m+1}[(\frac{1-s_0}{\bar{s}_0})^{m+1}N^{2\sigma_0-1}-1]|$$

$$-\frac{N^{-(1-\sigma_0)}}{\Phi(M+1,|s_0|\log N)}|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}|$$

$$-\frac{1}{\Phi(M+1,|s_0|\log N)}|(\frac{1-\bar{s}_0}{\bar{s}_0}N^{2\sigma_0-1}-\frac{s_0}{1-s_0})|\}$$

$$=\Phi(M+1,|1-s_0|\log N)N^{1-\sigma_0}|\int_{\sigma_0}^{1-\sigma_0}(\frac{|1-s_0|}{1-s})^{M+1}\frac{N^{\sigma_0-\sigma}}{1-s}d\sigma$$

$$-\sum_{m=0}^{M}\frac{1}{m+1}\frac{\Phi(m+1,|1-s_0|\log N)}{\Phi(M+1,|1-s_0|\log N)}(\frac{|1-s_0|}{1-s_0})^{m+1}[(\frac{1-s_0}{\bar{s}_0})^{m+1}N^{2\sigma_0-1}-1]|\geq 0$$

where when $s_0\neq 1-\bar{s}_0$, there are

$$|\int_{\sigma_0}^{1-\sigma_0}(\frac{|1-s_0|}{1-s})^{M+1}\frac{N^{\sigma_0-\sigma}}{1-s}d\sigma$$

$$-\sum_{m=0}^{M}\frac{1}{m+1}\frac{\Phi(m+1,|1-s_0|\log N)}{\Phi(M+1,|1-s_0|\log N)}(\frac{|1-s_0|}{1-s_0})^{m+1}[(\frac{1-s_0}{\bar{s}_0})^{m+1}N^{2\sigma_0-1}-1]|>0,$$

$$\frac{1}{\Phi(M+1,|s_0|\log N)}|(\frac{1-\bar{s}_0}{\bar{s}_0}N^{2\sigma_0-1}-\frac{s_0}{1-s_0})|=0$$

and by equations (7.10-7.11)

$$\frac{N^{-(1-\sigma_0)}}{\Phi(M+1,|s_0|\log N)}|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}|$$

$$\leq\frac{N^{-(1-\sigma_0)}}{\Phi(M+1,|s_0|\log N)}[2|1-\bar{s}_0|\frac{N^{1-\sigma_0}}{1-\sigma_0}+\frac{N^{1-\sigma_0}-N^{\sigma_0}}{\sigma_0\log N}]$$

$$=\frac{1}{\Phi(M+1,|s_0|\log N)}[\frac{2|1-\bar{s}_0|}{1-\sigma_0}+\frac{1-N^{2\sigma_0-1}}{\sigma_0\log N}]=0.$$

Thus, for $|s_0|>1$, we have

$$|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})|$$

$$\geq|\sum_{m=0}^{M}\frac{m!}{\log^{m+1}N}[\frac{N^{\bar{s}_0}}{\bar{s}_0^{m+1}}-\frac{N^{1-s_0}}{(1-s_0)^{m+1}}]-\frac{(M+1)!}{\log^{M+1}N}\int_{s_0}^{1-\bar{s}_0}\frac{N^{1-s}}{(1-s)^{M+2}}ds|$$

for $M+1\leq\log N\to\infty$ and

$$|\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}}-\frac{1-\bar{s}_0}{n^{1-\bar{s}_0}})+\sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s}+(\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0}-\frac{s_0}{1-s_0}N^{1-s_0})|$$

$$\leq |\sum_{m=0}^{M} \frac{m!}{\log^{m+1} N}[\frac{N^{\bar{s}_0}}{\bar{s}_0^{m+1}} - \frac{N^{1-s_0}}{(1-s_0)^{m+1}}] - \frac{(M+1)!}{\log^{M+1} N}\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+2}} ds|$$

for $M+1 \geq e|1-s_0|\log N \to \infty$.

Therefore, in the region $\log N < M+1 < e|1-s_0|\log N$ and for a plus ϵ and for $|s_0| > 1$, we can find numbers M, which satisfy

$$-\epsilon \leq |\sum_{n=1}^{N}(\frac{s_0}{n^{s_0}} - \frac{1-\bar{s}_0}{n^{1-\bar{s}_0}}) + \sum_{n=1}^{N}\int_{s_0}^{1-\bar{s}_0}\frac{ds}{n^s} + (\frac{1-\bar{s}_0}{\bar{s}_0}N^{\bar{s}_0} - \frac{s_0}{1-s_0}N^{1-s_0})|$$

$$-|\sum_{m=0}^{M} \frac{m!}{\log^{m+1} N}[\frac{N^{\bar{s}_0}}{\bar{s}_0^{m+1}} - \frac{N^{1-s_0}}{(1-s_0)^{m+1}}] - \frac{(M+1)!}{\log^{M+1} N}\int_{s_0}^{1-\bar{s}_0} \frac{N^{1-s}}{(1-s)^{M+2}} ds| \leq \epsilon.$$

Furthermore, since $\lim_{N\to\infty}\Delta(s_0, N)$ should exist, we can find a number N_ϵ so that when $N \geq N_\epsilon$ and $\log N < M+1 < e|1-s_0|\log N$ and $|s_0| > 1$, there is

$$0 \leq |\Delta(s_0, N)| \leq \epsilon.$$

Since ϵ can be arbitrarily small, for $|s_0| > 1$, we can obtain

$$|\Delta(s_0, N)| = 0.$$

Now, the curve integral expressed by Equation (6.6) can be estimated as

$$|\int_{s_0}^{1-\bar{s}_0}\zeta(s)ds| = |\int_{s_0}^{1-\bar{s}_0} s\int_0^\infty \frac{[x]-x}{x^{s+1}}dxds| = \lim_{N\to\infty}|\Delta(s_0, N)| = 0.$$

Since the integral

$$|\int_{s_0}^{1-\bar{s}_0}\zeta(s)ds| = |\int_{s_0}^{1-\bar{s}_0} s\int_0^\infty \frac{[x]-x}{x^{s+1}}dxds|$$

is a determined value and independent of the numbers N and M or the value $\frac{M+1}{\log N}$, thus, for $|s_0| > 1$, the curve integral (6.5) or (6.6) becomes

$$|\int_{s_0}^{1-\bar{s}_0}\zeta(s)ds| = |\int_{s_0}^{1-\bar{s}_0} s\int_0^\infty \frac{[x]-x}{x^{s+1}}dxds| = 0. \tag{7.12}$$

8. Lemma of central dissymmetry of $\zeta(s)$

Lemma 8.1. *The Riemann zeta function $\zeta(s) = u(\sigma, t) + iv(\sigma, t)$ is central unsymmetrical for the variable σ in the open region $(\sigma_0, 1-\sigma_0)$ about the point $(\sigma = \frac{1}{2}, t_0)$ along a horizontal line $t = t_0$ from s_0 to $1-\bar{s}_0$ except for the zeros of $\zeta(s) = 0$. This means that the following two equations*

$$u(\sigma, t_0) + u(1-\sigma, t_0) \equiv 0 \text{ and } v(\sigma, t_0) + v(1-\sigma, t_0) \equiv 0 \tag{8.1}$$

do not hold.

Proof. Let define functions

$$\eta_c(\sigma, t_0) = \int_0^\infty \frac{[x]-x}{x^{\frac{3}{2}+\sigma+it_0}}dx$$

and

$$\zeta_c(\sigma, t_0) = (\frac{1}{2} + \sigma + it_0)\int_0^\infty \frac{[x]-x}{x^{\frac{3}{2}+\sigma+it_0}}dx.$$

The function $\eta_c(\sigma, t_0)$ can be written as

$$\eta_c(\sigma, t_0) = u_\eta(\sigma, t_0) + iv_\eta(\sigma, t_0)$$

where

$$u_\eta(\sigma,t_0)=\int_0^\infty \frac{[x]-x}{x^{\frac{3}{2}+\sigma}} cos(t_0\log x)dx$$

and

$$v_\eta(\sigma,t_0)=-\int_0^\infty \frac{[x]-x}{x^{\frac{3}{2}+\sigma}} sin(t_0\log x)dx.$$

The function $\zeta_c(\sigma,t_0)$ can be written as

$$\begin{aligned}\zeta_c(\sigma,t_0)&=(\frac{1}{2}+\sigma+it_0)\eta_c(\sigma,t_0)\\&=(\frac{1}{2}+\sigma+it_0)[u_\eta(\sigma,t_0)+iv_\eta(\sigma,t_0)]\\&=u_\zeta(\sigma,t_0)+iv_\zeta(\sigma,t_0)\end{aligned}$$

where

$$u_\zeta(\sigma,t_0)=(\frac{1}{2}+\sigma)u_\eta(\sigma,t_0)-t_0v_\eta(\sigma,t_0)$$

and

$$v_\zeta(\sigma,t_0)=(\frac{1}{2}+\sigma)v_\eta(\sigma,t_0)+t_0u_\eta(\sigma,t_0).$$

By comparing $u_\zeta(\sigma,t_0)$ with $-u_\zeta(-\sigma,t_0)$ and $v_\zeta(\sigma,t_0)$ with $-v_\zeta(-\sigma,t_0)$, it is obvious that the following two equations

$$u_\zeta(\sigma,t_0)\equiv -u_\zeta(-\sigma,t_0)$$

and

$$v_\zeta(\sigma,t_0)\equiv -v_\zeta(-\sigma,t_0)$$

do not hold.

This means that the functions $u_\zeta(\sigma,t_0)$ and $v_\zeta(\sigma,t_0)$ are central unsymmetrical for the variable σ in the open region $-\frac{1}{2}<\sigma<\frac{1}{2}$ about the point $\sigma=0$ along a horizontal line $t=t_0$ from $-\sigma_0$ to σ_0 except for the zeros of $\zeta_c(\sigma,t_0)=0$.

Based upon Expression (5.1), when s varies along a horizontal line $t=t_0$ from s_0 to $1-\bar{s}_0$, there are

$$\zeta(s)=(\sigma+it_0)\int_0^\infty \frac{[x]-x}{x^{1+\sigma+it_0}}dx$$

and

$$\zeta(1-\bar{s})=(1-\sigma+it_0)\int_0^\infty \frac{[x]-x}{x^{2-\sigma+it_0}}dx.$$

By denoting $\sigma=\frac{1}{2}+\sigma'$ and substituting it into $\zeta(s)$ and $\zeta(1-\bar{s})$, we get

$$\zeta(s)=(\frac{1}{2}+\sigma'+it_0)\int_0^\infty \frac{[x]-x}{x^{\frac{3}{2}+\sigma'+it_0}}dx=\zeta_c(\sigma',t_0)$$

and

$$\zeta(1-\bar{s})=(\frac{1}{2}-\sigma'+it_0)\int_0^\infty \frac{[x]-x}{x^{\frac{3}{2}-\sigma'+it_0}}dx=\zeta_c(-\sigma',t_0).$$

Thus, for $\zeta(s)\neq 0$, the following two equations

$$u(\sigma,t_0)+u(1-\sigma,t_0)=u_\zeta(\sigma',t_0)+u_\zeta(-\sigma',t_0)\equiv 0$$

and

$$v(\sigma,t_0)+v(1-\sigma,t_0)=v_\zeta(\sigma',t_0)+v_\zeta(-\sigma',t_0)\equiv 0$$

do not hold.

The proof of the lemma is completed. $\square$

9. Proof of the Riemann hypothesis

Theorem 9.1. *The nontrivial zeros of the Riemann zeta function $\zeta(s)$ all have real part $\Re(s) = \frac{1}{2}$.*

Proof. Based upon the analysis of the curve integral (6.5) or (6.6) or (7.12) in the above sections, by denoting

$$\zeta(s) = u(\sigma, t) + iv(\sigma, t),$$

we get

$$Z(1-\bar{s}_0) - Z(s_0) = \int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = \int_{\sigma_0}^{1-\sigma_0} [u(\sigma, t_0) + iv(\sigma, t_0)]d\sigma = 0$$

for $|s_0| > 1$ and any pair of two different nontrivial zeros of the Riemann zeta function $\zeta(s)$, that is, the zeros $1 - \bar{s}_0$ and s_0 in the region $0 < \sigma < 1$.

Since the curve integral is only dependent of the start and end points of the curve, therefore, at least one of some equations has to be satisfied. These equations are

$$s_0 = 1 - \bar{s}_0$$

which means that σ_0 can be determined by $s_0 = 1 - \bar{s}_0$ since

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = \int_{s_0}^{s_0} \zeta(s)ds = 0,$$

$$u(\sigma, t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = i\int_{\sigma_0}^{1-\sigma_0} v(\sigma, t_0)d\sigma = if_v(\sigma_0) = 0,$$

$$v(\sigma, t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = \int_{\sigma_0}^{1-\sigma_0} u(\sigma, t_0)d\sigma = f_u(\sigma_0) = 0,$$

$$u(\sigma, t_0) = v(\sigma, t_0)$$

which means that σ_0 can be determined by

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = (1+i)\int_{\sigma_0}^{1-\sigma_0} u(\sigma, t_0)d\sigma = (1+i)f_u(\sigma_0) = 0,$$

$$u(\sigma, t_0) + u(1-\sigma, t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{\sigma_0}^{1-\sigma_0} u(\sigma, t_0)d\sigma = \int_{\sigma_0}^{1/2} [u(\sigma, t_0) + u(1-\sigma, t_0)]d\sigma \equiv 0$$

and

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = i\int_{\sigma_0}^{1-\sigma_0} v(\sigma, t_0)d\sigma = if_v(\sigma_0) = 0,$$

and

$$v(\sigma, t_0) + v(1-\sigma, t_0) \equiv 0$$

which means that σ_0 can be determined by

$$\int_{\sigma_0}^{1-\sigma_0} v(\sigma,t_0)d\sigma = \int_{\sigma_0}^{1/2} [v(\sigma,t_0)+v(1-\sigma,t_0)]d\sigma \equiv 0$$

and

$$\int_{s_0}^{1-\bar{s}_0} \zeta(s)ds = \int_{\sigma_0}^{1-\sigma_0} u(\sigma,t_0)d\sigma = f_u(\sigma_0) = 0$$

along a horizontal line $t = t_0$ from s_0 to $1-\bar{s}_0$ because of the continuity of the Riemann zeta function $\zeta(s)$, which means that $\zeta(s)$ must be central symmetric about the point $\sigma = \frac{1}{2}$ on the horizontal line.

First, $u(\sigma,t_0)$ and $v(\sigma,t_0)$ are not always equal to zero along a horizontal line from s to $1-\bar{s}$.

Next, it is obvious that there is $u(\sigma,t_0) \neq v(\sigma,t_0)$.

Thirdly, by Lemma (8.1), the following two equations

$$u(\sigma,t_0)+u(1-\sigma,t_0) \equiv 0$$

and

$$v(\sigma,t_0)+v(1-\sigma,t_0) \equiv 0$$

do not hold for a fixed value $t = t_0$. Thus, there is no value of σ_0 in the region $0 < \sigma_0 < 1$ except for $\sigma_0 = \frac{1}{2}$, which satisfies a group of the two different equations

$$f_u(\sigma_0) = \int_{\sigma_0}^{1-\sigma_0} u(\sigma,t_0)d\sigma = 0$$

and

$$f_v(\sigma_0) = \int_{\sigma_0}^{1-\sigma_0} v(\sigma,t_0)d\sigma = 0.$$

When the equations $u(\sigma,t_0) \equiv 0$, $v(\sigma,t_0) \equiv 0$, $u(\sigma,t_0) = v(\sigma,t_0)$, $u(\sigma,t_0)+u(1-\sigma,t_0) = 0$ and $v(\sigma,t_0)+v(1-\sigma,t_0) = 0$ are not satisfied, therefore, the remained equation

$$s_0 = 1-\bar{s}_0$$

must be satisfied, that is, there must be

$$\sigma_0 + it_0 = 1-\sigma_0+it_0.$$

By this equation, we obtain

$$\sigma_0 = \frac{1}{2}.$$

Since s_0 is any nontrivial zero of the Riemann zeta function $\zeta(s)$, there must be

$$\Re(s) = \frac{1}{2}$$

for any nontrivial zero of the Riemann zeta function $\zeta(s)$ and for $|s_0| > 1$.

For the limited region $|s_0| \leq 1$, we can calculate the nontrivial zeros of the Riemann zeta function $\zeta(s)$ to verify whether they have real part $\Re(s) = \frac{1}{2}$. Actually, computations of the nontrivial zeros of $\zeta(s)$ have been made in a region $[-T,T]$ for their imaginary parts, where $T \cong 5 \times 10^8$. Results of these computations show that the nontrivial zeros of the Riemann zeta function $\zeta(s)$ in the region $[-T,T]$ have real part $\Re(s) = \frac{1}{2}$. The Riemann hypothesis was computationally tested and found to be true for the first zeros by Brent et al. (1982), covering zeros in the region $0 < t < 81702130.19$. A computation made by Gourdon (2004) verifies that the Riemann hypothesis is true at least for all less than 2.4 trillion.

This completes the proof of the theorem and the Riemann hypothesis is proved. □

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